REPRESENTATION THEORETIC RELATIONS BETWEEN SCHUR POLYNOMIALS

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Dedicated to Prof. Dr. D.P. Zhelobenko at the occasion of his seventieth birthday

1. Introduction

In this paper we study level one integrable highest weight representations of some Kac–Moody Lie algebras $\underline{g}^{(k)}$, and the corresponding vertex operator constructions. We use the technique of fermionic operators, which is developed in e.g. [1] and [2].

A Heisenberg subalgebra (HSA) $\hat{s}$ plays an important role for the vertex operator representation of a Lie algebra $\underline{g}^{(k)}$. A HSA is an algebra on a basis $\{p_i, q_i\}_{i\in\mathbb{Z}_{>0}}$ and the canonical central element c with commutation relations $[p_i, q_i] = \delta_{ij}c$. If V is a representation of $\underline{g}^{(k)}$ such that for all $v \in V$ there exists an N such that $p_{i_1}\dots p_{i_l}v = 0$ whenever $i_1 + \dots + i_l > N$, it is completely reducible with respect to the action of $\hat{s}$. The only irreducible $\hat{s}$–module satisfying this condition is the ring of polynomials $\mathbb{C}[x]$ with the assignments $p_i \to \partial/\partial x_i$, $q_i \to x_i$, $c \to 1$. Therefore V can be identified with $V^+ \otimes \mathbb{C}[x]$ where the vacuum space V^+ is defined by the set of all vectors which are annihilated by the p_i.

In this way one obtains for every HSA $\hat{s}$ a realization for a given $\underline{g}^{(k)}$–module V. Such realizations can look very different, which is exemplified by the principal and homogeneous realization of the basic representation of the simplest affine Lie algebra $\widehat{sl}_2$.

This paper is organized as follows: In section 2 we recall the prerequisites about the Lie algebras A_∞ and $\widehat{gl}_n$ and their representations on the infinite wedge space that will be needed in the sequel. Each choice of a HSA leads to a decomposition $V \simeq \mathbb{C}[x]\otimes V^+$. In section 3 we look at several of those decompositions. In a number of cases, the main one being the partition $\underline{n} = \{n_1, n_2\}$, we compute directly the isomorphism of linear spaces $\gamma : \mathbb{C}[x^{(i)}]\otimes V^+ \longrightarrow \mathbb{C}[x]$, where $\mathbb{C}[x]$ corresponds with the principal realization. This gives relations between Schur polynomials. Further we look at the principal degree of the image of a polynomial which culminates in a q-dimension formula.

2. Prerequisites

2.1. **The Lie algebra $\overline{gl_\infty}$.** The associative Lie algebra gl_∞ is the collection of $\mathbb{Z}\times\mathbb{Z}$-matrices defined by

$$(1)\qquad gl_\infty = \{(a_{ij})_{i,j\in\mathbb{Z}} \mid a_{ij} \in \mathbb{C},\ \text{all but a finite number of the } a_{ij} \text{ are } 0\}$$

with the matrix commutator $[A, B] = AB - BA$ as Lie bracket. Denote by $\mathcal{E}_{ij}$ the matrix with 1 as the $(i, j)^{\text{th}}$ entry and all other entries equal to zero. The $\mathcal{E}_{ij}$ form a basis for gl_∞. Let $\mathbb{C}^\infty = \oplus_{j\in\mathbb{Z}}\mathbb{C}v(j)$ be an infinite dimensional complex vector space with basis $\{v(j) \mid j \in \mathbb{Z}\}$.

The Lie algebra gl_∞ acts on $\mathbb{C}^\infty$ by

(2) $$\mathcal{E}_{ij}v(k) = \delta_{jk}v(i)$$

The group GL_∞ associated with the Lie algebra gl_∞ is defined as

(3) $$GL_\infty = \{\mathrm{Id} + A \mid A \in gl_\infty, \det(\mathrm{Id} + A) \neq 0\}$$

In order to include the infinitesimal generators of certain flows one passes to the extension $\overline{gl_\infty}$ of the Lie algebra gl_∞. It is given by:

(4) $$\overline{gl_\infty} = \{(a_{ij})_{i,j\in\mathbb{Z}} \mid a_{ij} = 0 \text{ for } |i-j| \gg 0\}$$

Matrices in $\overline{gl_\infty}$ have a finite number of nonzero diagonals. The product of two matrices in $\overline{gl_\infty}$ is well defined, and is again in $\overline{gl_\infty}$, so $\overline{gl_\infty}$ is a Lie algebra containing gl_∞ as a subalgebra. The gl_∞-action on $\mathbb{C}^\infty$ extends naturally to an action of $\overline{gl_\infty}$.

2.2. **The infinite wedge space.** Let $\wedge^\infty\mathbb{C}^\infty$ be the vector space with a basis consisting of all semi-infinite exterior products of the basis elements $v(k)$ of $\mathbb{C}^\infty$ of the form:

(5) $$v(i_0) \wedge v(i_{-1}) \wedge v(i_{-2}) \wedge \ldots$$

such that $i_0 > i_{-1} > i_{-2} > \ldots$ and $i_{-l-1} = i_{-l} - 1$ for $l \gg 0$. The space $\wedge^\infty\mathbb{C}^\infty$ is called the infinite wedge space.

One can distinguish the basis elements (5) by their behaviour at large l. An element of the form (5) has charge k if $i_{-l} = k - l$ for all $l \gg 0$. For instance the vector

$$|k\rangle := v(k) \wedge v(k-1) \wedge v(k-2) \wedge \ldots$$

has charge k. The vector $|k\rangle$ is called the k^{th} vacuum. The vector space of all vectors of charge k is denoted by $F^{(k)}$. One has a decomposition of the infinite wedge space in sectors of fixed charge:

$$\wedge^\infty\mathbb{C}^\infty = \bigoplus_{k\in\mathbb{Z}} F^{(k)}$$

For every $k \in \mathbb{Z}$ one defines linear operators $\psi(k)$ and $\psi^*(k)$ on the infinite wedge space by their action on the basis vectors:

(6) $$\begin{aligned} \psi(k)\,(v(i_0) \wedge v(i_{-1}) \wedge v(i_{-2}) \wedge \ldots) &= v(k) \wedge v(i_0) \wedge v(i_{-1}) \wedge v(i_{-2}) \wedge \ldots \\ \psi^*(k)\,(v(i_0) \wedge v(i_{-1}) \wedge v(i_{-2}) \wedge \ldots) &= \sum_{l=0}^{\infty}(-1)^l \delta_{k,i_{-l}} v(i_0) \wedge v(i_{-1}) \wedge \ldots \wedge \widehat{v}(i_{-l}) \wedge \ldots \end{aligned}$$

where the notation $\widehat{v}(i_{-l})$ means that the vector $v(i_{-l})$ is deleted. These operators satisfy the anticommutation relations:

(7) $$\{\psi(k), \psi(l)\} = 0 = \{\psi^*(k), \psi^*(l)\} \text{ and } \{\psi(k), \psi^*(l)\} = \delta_{kl}$$

where the anticommutator $\{A, B\}$ is defined by $\{A, B\} := AB + BA$.

Any element of the infinite wedge space $\wedge^\infty\mathbb{C}^\infty$ can be written as a finite linear combination of elements of the form

(8) $$\psi(k_1)\cdots\psi(k_r)\psi^*(l_1)\cdots\psi^*(l_s)|0\rangle$$

where $k_1 > \ldots > k_r > 0 \geq l_1 > \ldots > l_s$. This means that one could also have constructed the space $\wedge^\infty\mathbb{C}^\infty$ in a different manner. Namely, let Cl be the Clifford algebra on generators

$\psi(i)$, $\psi^*(i)$ $i \in \mathbb{Z}$ with relations (7). Define the socalled fermionic Fock space F as the unique irreducible Cl-module, which admits a vacuum vector $|0\rangle$ such that

(9) $$\psi(i)|0\rangle = 0 \qquad \forall\, i \leq 0 \text{ and } \psi^*(i)|0\rangle = 0 \qquad \forall\, i > 0$$

Then we have $F = \wedge^\infty\mathbb{C}^\infty$. The fermionic Fock space F is also called the spin representation of Cl.

The space $\wedge^\infty\mathbb{C}^\infty$ can be equiped with an inner product (,), which is uniquely determined by the requirements

$$(|0\rangle, |0\rangle) = 1 \text{ and } \psi(k)^\dagger = \psi^*(k),$$

where $A^\dagger$ denotes the adjoint of a linear operator A on $\wedge^\infty\mathbb{C}^\infty$ w.r.t. the inner product (,). Then the elements (8) have length 1. One defines

(10) $$\langle 0|A|0\rangle := (|0\rangle, A|0\rangle)$$

where A is a linear operator on $\wedge^\infty\mathbb{C}^\infty$. The quantity (10) is called the vacuum expectation value of A. Sometimes one abbreviates it to $\langle A\rangle$.

2.3. **Representations of gl_∞ on $\wedge^\infty\mathbb{C}^\infty$.** One can define representations ρ of gl_∞ and the corresponding one R of GL_∞ on $\wedge^\infty\mathbb{C}^\infty$ by

(11) $$\begin{aligned}\rho(a)(v(i_0)\wedge v(i_{-1})\wedge v(i_{-2})\wedge\ldots) \;:=\;& av(i_0)\wedge v(i_{-1})\wedge v(i_{-2})\wedge\ldots\\ &+v(i_0)\wedge av(i_{-1})\wedge v(i_{-2})\wedge\ldots\\ &+v(i_0)\wedge v(i_{-1})\wedge av(i_{-2})\wedge\ldots+\ldots\end{aligned}$$

(12) $$R(A)(v(i_0)\wedge v(i_{-1})\wedge v(i_{-2})\wedge\ldots) := Av(i_0)\wedge Av(i_{-1})\wedge Av(i_{-2})\wedge\ldots$$

The action of the elements $\mathcal{E}_{ij}$ can be written as $\rho(\mathcal{E}_{ij}) = \psi(i)\psi^*(j)$.

The submodule $F^{(k)}$ is an irreducible highest weight module for the algebra gl_∞. One has for $j > i$

(13) $$\rho(\mathcal{E}_{ij})|k\rangle = 0$$

and

(14) $$\rho(\mathcal{E}_{ii})|k\rangle = \theta_k(\mathcal{E}_{ii})|k\rangle$$

where the linear mapping $\theta_k : \oplus_{i\in\mathbb{Z}}\mathbb{C}\mathcal{E}_{ii} \to \mathbb{C}$ is defined by

(15) $$\theta_k(\mathcal{E}_{ii}) = \begin{cases} 0 \text{ if } i > k \\ 1 \text{ if } i \leq k \end{cases}$$

It is not possible to extend the representation ρ to the extension $\overline{gl_\infty}$ by linearity. For example the identity matrix Id is an element of $\overline{gl_\infty}$. Its action on the vacuum vector $|0\rangle$ is $\rho(\text{Id})|0\rangle = \infty|0\rangle$, so it is not well defined. A remedy for this problem would be to subtract a term from ρ and to define

(16) $$\pi(\mathcal{E}_{ij}) := \rho(\mathcal{E}_{ij}) - \delta_{ij}\theta_0(\mathcal{E}_{ii})I.$$

For each $A = \sum_{i,j} \alpha ij \mathcal{E}_{ij} \in \overline{gl_\infty}$ the operator $\pi(A) = \sum_{i,j} \alpha ij \pi(\mathcal{E}_{ij})$ is well-defined. However, it is not a representation of $\overline{gl_\infty}$ anymore since

(17)
$$\begin{aligned}[\pi(\mathcal{E}_{ij}), \pi(\mathcal{E}_{kl})] &= [\rho(\mathcal{E}_{ij}), \rho(\mathcal{E}_{kl})] \\ &= \delta_{jk}\rho(\mathcal{E}_{il}) - \delta_{li}\rho(\mathcal{E}_{kj}) \\ &= \delta_{jk}\pi(\mathcal{E}_{il}) - \delta_{li}\pi(\mathcal{E}_{kj}) + \delta_{jk}\delta_{li}\theta_0(\mathcal{E}_{ii} - \mathcal{E}_{jj}) \\ &= \pi([\mathcal{E}_{ij}, \mathcal{E}_{kl}]) + \delta_{jk}\delta_{li}\theta_0(\mathcal{E}_{ii} - \mathcal{E}_{jj}).\end{aligned}$$

This additional term determines a 2-cocycle $\mu : \overline{gl_\infty} \times \overline{gl_\infty} \to \mathbb{C}$ by the bilinear extension of

(18)
$$\mu(\mathcal{E}_{ij}, \mathcal{E}_{kl}) := \delta_{jk}\delta_{li}\theta_0(\mathcal{E}_{ii} - \mathcal{E}_{jj}).$$

The 2-cocycle μ determines a central extension $A_\infty := \overline{gl_\infty} \oplus \mathbb{C}c$ of $\overline{gl_\infty}$ with the Lie bracket on A_∞ given by

(19)
$$[A + \alpha c, B + \beta c] = AB - BA + \mu(A, B)c \qquad A, B \in \overline{gl_\infty},\ \alpha, \beta \in \mathbb{C}.$$

If one defines $\pi(A + \alpha c) = \pi(A) + \alpha\,\mathrm{Id}$ then π is a so-called $c = 1$ faithful representation of A_∞ and one writes $\underline{g}_A := \pi(A_\infty)$. $\underline{g}_A$.

The representation π can be expressed in terms of the fermions ψ and ψ^*. One has $\pi(\mathcal{E}_{ij}) =: \psi(i)\psi^*(j) :$, where the normal ordering $: \psi(i)\psi^*(j) :$ is defined by

(20)
$$\begin{aligned}: \psi(i)\psi^*(j) : &:= \psi(i)\psi^*(j) - \langle 0|\psi(i)\psi^*(j)|0\rangle \\ &= \begin{cases}\psi(i)\psi^*(j) \text{ if } j > 0 \\ -\psi^*(j)\psi(i) \text{ otherwise}\end{cases}\end{aligned}$$

2.4. **The principal degree.** On $F^{(0)}$ one assigns a degree to a monomial of the form (5) by

(21)
$$\deg(v(i_0) \wedge v(i_{-1}) \wedge v(i_{-2}) \wedge \ldots) = \sum_{s=0}^{\infty}(i_{-s} + s)$$

Then the degree is a finite nonnegative integer. This degree is called the principal degree. Let $F_l^{(0)}$ denote the linear span of all vectors in $F^{(0)}$ of degree l. Then

$$F^{(0)} = \bigoplus_{l \geq 0} F_l^{(0)} \text{ and } \dim_q F^{(0)} := \sum_l (\dim F_l^{(0)})q^l = \varphi(q)^{-1},$$

where $\varphi(q) = \prod_{k>0}(1 - q^k)$.

One can express the degree in terms of the $\psi(i)$ and the $\psi^*(j)$. If one put $\deg\psi(i) = i$, $\deg\psi^*(j) = -j$ and $\deg(|0\rangle) = 0$,then the degree of (8) is given by $k_1 + \ldots + k_r - l_1 - \ldots - l_s$. This degree coincides with the degree above on $F^{(0)}$. Define

(22)
$$H_0 = \sum_{k \in \mathbb{Z}} k : \psi(k)\psi^*(k) :$$

Then $[H_0, \psi(k)] = k\psi(k)$, $[H_0, \psi^*(k)] = -k\psi^*(k)$ and $H_0|0\rangle = 0$. The operator H_0 is called the Hamiltonian or Energy operator. Its eigenvalues are the degrees of the eigenvectors.

2.5. **The oscillator algebra.** Define the shift operators $M_k : \mathbb{C}^\infty \mapsto \mathbb{C}^\infty$ by $M_k v(j) = v(j-k)$. Then the corresponding matrices $\Lambda_k \in \overline{gl_\infty}$ are

(23) $$\Lambda_k = \sum_{j\in\mathbb{Z}} \mathcal{E}_{j,j+k}$$

They form a commutative subalgebra of $\overline{gl_\infty}$. It is a straightforward verification that $\mu(\Lambda_k, \Lambda_l) = k\delta_{k+l,0}$ so the Λ_k have in A_∞ the Lie bracket

(24) $$[\Lambda_k, \Lambda_l] = k\delta_{k+l,0}c$$

Then $(\oplus_{k\in\mathbb{Z}}\mathbb{C}\Lambda_k) \oplus \mathbb{C}c$ is a subalgebra of A_∞, the so-called bosonic oscillator algebra $\mathcal{A}$.

Define $\alpha(k) := \pi(\Lambda_k)$, then

(25) $$\alpha(k) = \sum_{j\in\mathbb{Z}} :\psi(j)\psi^*(j+k):$$

They satisfy $[\alpha(k), \alpha(l)] = k\delta_{k+l,0}$. The operator $\alpha(0)$ is called the charge operator, its eigenvalues are the charges: $\alpha(0)|_{F^{(k)}} = k \cdot I$. There holds $\alpha(k)|0\rangle = 0$ if $k > 0$. From $[H_0, \alpha(k)] = -k\alpha(k)$ one sees that $\alpha(k)$ has principal degree $-k$.

There exists a standard representation of $\mathcal{A}$ in the space of polynomials in infinitely many variables x_k $(k \geq 1)$. It is given by:

$$\alpha(k) \longrightarrow \frac{\partial}{\partial x_k}, \alpha(-k) \longrightarrow kx_k, \text{ for } k \geq 1, \ \alpha(0) \longrightarrow \mu\,\mathrm{Id}, \ c \longrightarrow \mathrm{Id}$$

In this space $\mathbb{C}[x] = \mathbb{C}[x_1, x_2, x_3, \ldots]$ one has

(26) $$\deg(x_k) = k$$

2.6. **Vertex operators.** One defines $\psi(z) := \sum \psi(k)z^k$, $\psi^*(z) := \sum \psi^*(k)z^{-k}$, where z is a formal parameter. The $\psi(z)$, $\psi^*(z)$ are generating operators for the $\psi(k)$, $\psi^*(k)$. They are called fermionic fields. These fields can be expressed in terms of the bosons $\alpha(k)$. The fields are eigenvectors for the adjoint action of the $\alpha(k)$:

$$[\alpha(k), \psi(z)] = z^k\psi(z) \text{ and } [\alpha(k), \psi^*(z)] = -z^k\psi^*(z)$$

There is a well-known expression for the fermions in terms of the bosons (see e.g. [4]):

(27) $$\begin{aligned} \psi(z) &= Qz^{\alpha(0)+1}E^-(z)E^+(z) \\ \psi^*(z) &= Q^{-1}z^{-\alpha(0)}E^-(z)^{-1}E^+(z)^{-1}, \end{aligned}$$

where

$$E^-(z) = \exp\left(-\sum_{k<0}\frac{1}{k}\alpha(k)z^{-k}\right) \text{ and } E^+(z) = \exp\left(-\sum_{k>0}\frac{1}{k}\alpha(k)z^{-k}\right)$$

and $Q : F^{(k)} \to F^{(k+1)}$ is an operator satisfying

(28) $$\begin{aligned} Q\psi(z) &= z^{-1}\psi(z)Q \\ Q\psi^*(z) &= z\psi^*(z)Q \end{aligned} \quad \text{and} \quad \begin{aligned} Q|0\rangle &= \psi(1)|0\rangle \\ Q^{-1}|0\rangle &= \psi^*(0)|0\rangle \end{aligned}$$

The operator Q commutes with all bosons $\alpha(k)$ except for $k = 0$:

(29) $$[\alpha(k), Q] = \delta_{k0}Q$$

One can look at the generating operator for the action of A_∞ on $F^{(k)}$:

$$X(u,v) := \sum_{k,l\in\mathbb{Z}} \pi(\mathcal{E}_{kl})u^k v^{-l} = \sum_{k,l\in\mathbb{Z}} :\psi(k)\psi^*(l): u^k v^{-l} =: \psi(u)\psi^*(v): \tag{30}$$

Now one uses equations (27) and $:\psi(u)\psi^*(v) := \psi(u)\psi^*(v) - \langle 0|\psi(u)\psi^*(v)|0\rangle$ and obtain

$$X(u,v) = \frac{(v/u)^{-\alpha(0)}}{1-v/u} E^-(u)E^-(v)^{-1}E^+(u)E^+(v)^{-1} - \frac{1}{1-v/u}I \tag{31}$$

$$= \frac{(v/u)^{-\alpha(0)}}{1-v/u} \exp\left(\sum_{k>0}\frac{1}{k}(u^k - v^k)\alpha(-k)\right) \exp\left(-\sum_{k>0}\frac{1}{k}(u^{-k} - v^{-k})\alpha(k)\right) - \frac{1}{1-v/u}I \tag{32}$$

Here $\frac{1}{1-v/u}$ is a formal power series in u and v: $\frac{1}{1-v/u} := \sum_{k\geq 0}(v/u)^k$. Thus one sees that the action of the algebra A_∞ on the infinite wedge space can be completely expressed in terms of the action of the subalgebra $\mathcal{A}$ of all oscillators. Combining this with the fact that the charge k sector is an irreducible A_∞-module, one concludes that $F^{(k)}$ must remain irreducible under the action of this oscillator algebra.

2.7. **Schur polynomials.** The elementary Schur polynomials $S_k(x) \in \mathbb{C}[x]$ are defined by the generating function

$$\sum_{k\in\mathbb{Z}} S_k(x)z^k = \exp\left(\sum_{k>0} x_k z^k\right) \tag{33}$$

Then

$$S_k(x) = 0 \quad \text{for } k < 0, \qquad S_0(x) = 1 \tag{34}$$

$$S_k(x) = \sum_{k_1+2k_2+\ldots=k} \frac{x_1^{k_1}}{k_1!}\frac{x_2^{k_2}}{k_2!}\cdots \quad \text{for } k > 0 \tag{35}$$

One denotes the set of partitions by Par. Thus $\lambda \in Par$ is a nonincreasing finite sequence of positive integers $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_r > 0$. In the sequel also a different notation will be used: the integers are labeled by $k \leq 0$, and the sequence is extended with zeros; so we have

$$Par \simeq \{(\lambda_k) \mid k \leq 0,\ \exists N : p > N \Rightarrow \lambda_{-p} = 0\}.$$

To each $\lambda \in Par$ one associates the Schur polynomial $S_\lambda(x)$ defined by the determinant

$$S_\lambda(x) := \det(S_{\lambda_i - i + j}(x)) \tag{36}$$

With respect to the principal gradation on $\Lambda^\infty\mathbb{C}^\infty$ the Schur polynomial $S_\lambda(x)$ is a homogeneous polynomial of degree $|\lambda| := \lambda_1 + \lambda_2 + \ldots$. The $(S_\lambda)_{\lambda\in Par}$ form a basis of $\mathbb{C}[x]$.

2.8. **Boson–fermion correspondence.** One has an isomorphism of $\mathcal{A}$-modules

(37) $$\mathbb{C}[x] = \mathbb{C}[x_1, x_2, \ldots] \simeq \{\alpha(-k_1)\ldots\alpha(-k_r)|0\rangle \mid k_i > 0\} = F^{(0)}$$

The isomorphism $\sigma : F^{(0)} \to \mathbb{C}[x]$ is called the boson-fermion correspondence.

Every $F^{(k)}$ is isomorphic with $\mathbb{C}[x]$. The factor Q^k indicates the charge sector $F^{(k)}$. Then $\sigma : \Lambda^\infty\mathbb{C}^\infty \to \mathbb{C}[x; Q, Q^{-1}]$ is characterized by ($k \in \mathbb{Z}$, $l > 0$)

$$\sigma(|k\rangle) = Q^k, \sigma\alpha(l)\sigma^{-1} = \frac{\partial}{\partial x_l}, \sigma\alpha(-l)\sigma^{-1} = lx_l, \sigma\alpha(0)\sigma^{-1} = Q\frac{\mathrm{d}}{\mathrm{dQ}}, \sigma\,\mathrm{Id}\,\sigma^{-1} = \mathrm{Id}\,.$$

Define

(38) $$H(x) := \sum_{k>0} x_k\alpha(k)$$

Then $e^{H(x)}$ is well defined on $\Lambda^\infty\mathbb{C}^\infty$. On the charge zero sector σ is given by (see [2] or [1])

(39) $$\sigma(A|0\rangle) = \langle 0|e^{H(x)}A|0\rangle$$

In general it is given by

(40) $$\sigma(A|0\rangle) = \sum_{k\in\mathbb{Z}} z^k\langle k|e^{H(x)}A|0\rangle = \sum_{k\in\mathbb{Z}} z^k\sigma_k(A|0\rangle)$$

One has the following expressions for the transported operators

$$\Gamma_+(z) \;:=\; \sigma E^+(z)\sigma^{-1} \;=\; \exp\left(\sum_{k>0}\frac{1}{k}z^{-k}\frac{\partial}{\partial x_k}\right)$$

$$\Gamma_-(z) \;:=\; \sigma E^-(z)\sigma^{-1} \;=\; \exp\left(\sum_{k>0} z^k x_k\right)$$

and

$$\begin{aligned}\Gamma(u,v) \;&:=\; \sigma E^-(u)E^-(v)^{-1}E^+(u)E^+(v)^{-1}\sigma^{-1}\\ &=\; \exp\left(\sum_{k>0}(u^k - v^k)x_k\right)\exp\left(-\sum_{k>0}\frac{1}{k}(u^{-k} - v^{-k})\frac{\partial}{\partial x_k}\right)\end{aligned}$$

In [3] the following theorem is proved:

Theorem 2.9. *For $v(i_0)\wedge v(i_{-1})\wedge v(i_{-2})\wedge\ldots \in F^{(0)}$ there holds*

(41) $$\sigma(v(i_0)\wedge v(i_{-1})\wedge v(i_{-2})\wedge\ldots) = S_{i_0,i_{-1}+1,i_{-2}+2,\ldots}(x).$$

2.10. **The Kac–Moody algebra $\widehat{gl}_n$.** Let gl_n denote the Lie algebra of all complex $n\times n$-matrices with the standard basis $(E_{ij})_{1\le i,j\le n}$. One defines the loop algebra $\widetilde{gl}_n$ as

(42) $$\widetilde{gl}_n := \bigoplus_{k\in\mathbb{Z}} t^k gl_n$$

with commutation relations $[At^k, Bt^l] = [A,B]t^{k+l}$. The loop algebra $\widetilde{gl}_n$ acts in a natural way on $\mathbb{C}[t,t^{-1}]^n$. Let $(e_i)_{1\le i\le n}$ be the standard basis of $\mathbb{C}^n$. Define $u_{nk+j} := t^{-k}e_j$. Then the $(u_k)_{k\in\mathbb{Z}}$ form a basis of $\mathbb{C}[t,t^{-1}]^n$ over $\mathbb{C}$. Thus $\mathbb{C}[t,t^{-1}]^n \simeq \mathbb{C}^\infty$. The Lie algebra $\widetilde{gl}_n$ embeds into $\overline{gl_\infty}$ by the Lie algebra homomorphism $\iota : \widetilde{gl}_n \to \overline{gl_\infty}$ determined by

$$\iota(t^k E_{ij}) = \sum_{s\in\mathbb{Z}} \mathcal{E}_{n(s-k)+i,ns+j} \tag{43}$$

The image of $\widetilde{gl}_n$ in $\overline{gl_\infty}$ consists of

$$\iota(\widetilde{gl}_n) = \{(a_{ij}) \in \overline{gl_\infty} \mid a_{i+n,j+n} = a_{i,j}\}.$$

By restricting the 2-cocycle μ to $\iota(\widetilde{gl}_n)$ one gets a 2-cocycle for $\widetilde{gl}_n$. It is given by

$$\mu\big(\iota(t^k A), \iota(t^l B)\big) = k\delta_{k+l,0}\mathrm{Tr}(AB) =: k\delta_{k+l,0}(A|B)$$

and determines a central extension $\widehat{gl}_n$ of $\widetilde{gl}_n$. This Lie algebra is called the affine Kac–Moody algebra associated to gl_n. It will be viewed as a Lie subalgebra of A_∞.

Define $E \in gl_n$ by $E := \sum_{k=1}^{n-1} E_{k,k+1} + E_{1n}$. Then

$$\underline{h}_p := \bigoplus_{1\le k\le n} \mathbb{C}E^k \tag{44}$$

is a Cartan subalgebra (CSA) of gl_n, the principal CSA. This E induces the element $\overline{E} = \sum_{k=1}^{n-1} E_{k,k+1} + E_{1n}t \in \widetilde{gl}_n$ (see e.g. [3]). It is easy to show that $\iota(\overline{E}^k) = \Lambda_k,\ k \in \mathbb{Z}$.

We now have a representation $\pi \circ \iota$ of $\widehat{gl}_n$ in $\wedge^\infty\mathbb{C}^\infty$. The representation $F^{(m)}$ is irreducible under $\widehat{gl}_n$ since the latter contains the Λ_j $(j \in \mathbb{Z})$ and $F^{(m)}$ is irreducible under the action of $\mathcal{A}$. The action of $\widehat{gl}_n$ in $F^{(0)}$ is given by the vertex operator

$$: \psi(\omega^{-k}u)\psi^*(\omega^{-l}u) : \qquad 1 \le k, l \le n \tag{45}$$

where $\omega = e^{\frac{2\pi i}{n}}$ (see [4]).

3. Isomorphisms of Schur polynomials

3.1. **Fermions with various components.** It is well-known that the conjugacy classes in $W(gl_n)$, the Weyl group of gl_n, are parametrized by partitions of n. Any partition $\underline{n}$ of the number n in s parts $n_1,\ n_2, \ldots,\ n_s$ determines a direct sum decomposition $\mathbb{C}^n \simeq \mathbb{C}^{n_1} \oplus \cdots \oplus \mathbb{C}^{n_s}$ and an associated block decomposition of a $n \times n$-matrix. The diagonal blocks correspond to Lie algebras gl_{n_i} and the principal construction of subsection 1.9 tells us how to make vertex operators describing the action of the affine algebra $\widehat{gl}_{n_i}$. So one just takes s copies of the construction above or, which is the same thing, one should work with s-component fermions $\psi_i(k),\ \psi_i^*(k),\ 1 \le i \le s,\ k \in \mathbb{Z}$. The problem is how to find vertex operators associated to the off diagonal blocks.

A partition $\underline{n}$ of n leads to s-component fermions:

$$\begin{aligned} \psi_i(l + mn_i) &:= \psi(n_1 + \ldots + n_{i-1} + l + mn) \\ \psi_i^*(l + mn_i) &:= \psi_i^*(n_1 + \ldots + n_{i-1} + l + mn) \qquad 1 \le l \le n_i, m \in \mathbb{Z} \end{aligned} \tag{46}$$

These fermions satisfy the relations

$$\begin{aligned} \{\psi_i(k), \psi_j(l)\} = \{\psi_i^*(k), \psi_j^*(l)\} &= 0 \\ \{\psi_i(k), \psi_j^*(l)\} &= \delta_{ij}\delta_{kl} \end{aligned} \tag{47}$$

In terms of these fermions the spin module $\wedge^\infty\mathbb{C}^\infty$ can also be defined as the unique Cl-module generated by a vacuum $|0\rangle$ satisfying

$$\psi_i(k)|0\rangle = 0, k \leq 0\ \forall i \text{ and } \psi_i^*(k)|0\rangle = 0, k > 0\ \forall i.$$

Now one defines

$$\alpha_i(k) := \sum_{l\in\mathbb{Z}} :\psi_i(l)\psi_i^*(l+k): \tag{48}$$

Then one obtains in the same way as in the one-component case

$$\psi_i(z) = Q_i z^{\alpha_i(0)+1}\exp\left(-\sum_{k<0}\frac{1}{k}\alpha_i(k)z^{-k}\right)\exp\left(-\sum_{k>0}\frac{1}{k}\alpha_i(k)z^{-k}\right) \tag{49}$$

$$\psi_i^*(z) = Q_i^{-1} z^{-\alpha_i(0)}\exp\left(\sum_{k<0}\frac{1}{k}\alpha_i(k)z^{-k}\right)\exp\left(\sum_{k>0}\frac{1}{k}\alpha_i(k)z^{-k}\right) \tag{50}$$

where the operators Q_i satisfy

$$\begin{aligned} Q_i\psi_i(k) &= \psi_i(k+1)Q_i \\ Q_i\psi_i^*(k) &= \psi_i^*(k+1)Q_i \\ Q_i\psi_j(k) &= -\psi_j(k)Q_i \qquad i\neq j \\ Q_i\psi_i^*(k) &= -\psi_i^*(k)Q_i \qquad i\neq j \\ Q_i|0\rangle &= \psi_i(1)|0\rangle \\ \{Q_i,Q_j\} &= 0 \qquad i\neq j \end{aligned} \tag{51}$$

The basic representation $L(\Lambda_0)$ of gl_∞ is isomorphic to $F^{(0)}$. The vacuum space V^+ of the gl_n-module $F^{(0)}$ is spanned by the vectors

$$T_1^{m_1}\cdots T_{n-1}^{m_{n-1}}|0\rangle, \quad m_i\in\mathbb{Z} \tag{52}$$

where $T_i := Q_iQ_{i+1}^{-1}$, $1\leq i\leq n-1$ (see [4]).

3.2. **Two types of fermions.** Let n be a positive integer. Here we look at partitions of n in two parts: $n = n_1 + n_2$ where $n_1, n_2 \in \mathbb{Z}_{>0}$. Let $(v(k))_{k\in\mathbb{Z}}$ be a basis of $\mathbb{C}^\infty$. We can relabel this basis with respect to the partition of n mentioned above. We then get:

$$\begin{aligned} v_1(mn_1+l) &= v(mn+l)\ m\in\mathbb{Z}, 1\leq l\leq n_1 \\ v_2(mn_2+l) &= v(mn+n_1+l)\ m\in\mathbb{Z}, 1\leq l\leq n_2 \end{aligned}$$

We can also relabel the fermionic operators $\psi(k)$ and $\psi^*(k)$ in the same way. These relabeled operators then have the anticommutation relations

$$\begin{aligned} \{\psi_i(k),\psi_j(l)\} = \{\psi_i^*(k),\psi_j^*(l)\} &= 0 \\ \{\psi_i(k),\psi_j^*(l)\} &= \delta_{ij}\delta_{kl}\ i,j=1,2 \end{aligned}$$

These are the anticommutation relations for fermionic operators of two different types. These relabeling leads to the following isomorphism

$$\mathbb{C}[x] \simeq F^{(0)} \simeq F_1^{(0)}\otimes F_2^{(0)}\otimes\mathbb{C}[T,T^{-1}] \simeq \mathbb{C}[x^{(1)},x^{(2)},T,T^{-1}]$$

Here T is the operator $Q_1Q_2^{-1}$. It creates a fermion of type 1 and annihilates a fermion of type 2.

It is convenient to define the following functions:

$$f_1(k) := \left\lfloor \frac{k-1}{n_1} \right\rfloor n_2 + k \text{ and } f_2(k) := \left\lfloor \frac{k-1}{n_2} \right\rfloor n_1 + n_1 + k.$$

Here is $k \in \mathbb{Z}$. These functions satisfy the relations $v_i(k) = v(f_i(k))$ where $i = 1, 2$. Denote by $\gamma : \mathbb{C}[x^{(1)}, x^{(2)}, T, T^{-1}] \to \mathbb{C}[x]$ the above mentioned isomorphism. A basis for $\mathbb{C}[x^{(1)}, x^{(2)}, T, T^{-1}]$ is given by $T^k S_\mu(x^{(1)}) S_\lambda(x^{(2)})$, where $\lambda, \mu \in Par$. We now want to compute the images of the basis elements in $\mathbb{C}[x]$. For $k \in \mathbb{Z}$ we define $k_i \in \mathbb{Z}, 1 \le k_i \le n_i$ by $k \equiv k_i \bmod n_i$. Then we are ready for the following

Theorem 3.3. *For $k > 0$, $k = mn_2 + k_2$ we have*

$$\gamma(T^k S_\mu(x^{(1)}) S_\lambda(x^{(2)})) = (-1)^{c_+} S_{\bar{\gamma}(k,\mu,\lambda)}(x) \tag{53}$$

where

$$c_+ = \frac{n}{2n_2}k^2 - \frac{1}{2}(n_1+1)k + \frac{n_1}{2n_2}k_2(n_2 - k_2)$$

For $-k - mn_1 + 1 \le r \le 0$ we have

$$\begin{aligned} \bar{\gamma}(r, \mu, \lambda) &= \left\lfloor \frac{\lambda_r + k + r - 1}{n_1} \right\rfloor n_2 + \lambda_r + k \\ &= f_1(\lambda_r + k + r) - r \end{aligned}$$

For $p < 0, 1 \le q \le n_1, pn + q \le -k - mn_1 = -mn - k_2$ we have

$$\begin{aligned} \bar{\gamma}(pn + q, \mu, \lambda) &= \left\lfloor \frac{\lambda_{-k+pn_1+q} + q - 1}{n_1} \right\rfloor n_2 + \lambda_{-k+pn_1+q} \\ &= f_1(\lambda_{-k+pn_1+q} + q) - q \end{aligned}$$

For $p < 0, 1 \le q \le n_2, pn + q + n_1 \le -k - mn_1 = -mn - k_2$ we have

$$\begin{aligned} \bar{\gamma}(pn + q + n_1, \mu, \lambda) &= \left\lfloor \frac{\mu_{k+pn_2+q} + q - 1}{n_2} \right\rfloor n_1 + \mu_{k+pn_2+q} \\ &= f_2(\mu_{k+pn_2+q} + q) - q - n_1 \end{aligned}$$

In the case that the power of T is negative we have ($k > 0, k = mn_1 + k_1$):

$$\gamma(T^{-k} S_\mu(x^{(1)}) S_\lambda(x^{(2)})) = (-1)^{c_-} S_{\bar{\gamma}(-k,\mu,\lambda)}(x)$$

where

$$c_- = \frac{n}{2n_1}k^2 + \frac{1}{2}(n_2 - 1)k + \frac{n_2}{2n_1}k_1(n_1 - k_1)$$

For $-mn - k_1 - n_2 + 1 \le r \le 0$ we have

$$\begin{aligned} \bar{\gamma}(r, \mu, \lambda) &= \left\lfloor \frac{\mu_r + k + r - 1}{n_2} \right\rfloor n_1 + \mu_r + k + n_1 \\ &= f_2(\mu_r + k + r) - r \end{aligned}$$

For $p < 0, 1 \le q \le n_1, pn + q \le -mn - n_2 - k_1$ we have

$$\begin{aligned} \bar{\gamma}(pn + q, \mu, \lambda) &= \left\lfloor \frac{\lambda_{k+pn_1+q} + q - 1}{n_1} \right\rfloor n_2 + \lambda_{k+pn_1+q} \\ &= f_1(\lambda_{k+pn_1+q} + q) - q \end{aligned}$$

For $p<0, 1\le q\le n_2, pn+q+n_1\le -mn-k_1-n_2=-(m+1)n-k_1+n_1$ *we have*

$$\begin{aligned}\bar\gamma(pn+q+n_1,\mu,\lambda) &= \left\lfloor\frac{\mu_{-k+pn_2+q}+q-1}{n_2}\right\rfloor n_1+\mu_{-k+pn_2+q}\\ &= f_2(\mu_{-k+pn_2+q}+q)-q-n_1\end{aligned}$$

These last formulas are also valid for $k=0$. *Then we have to take* $m=-1$ *and* $k_1=n_1$.

Proof. The formulas have been calculated following the isomorphism $\mathbb{C}[x^{(1)},x^{(2)},T,T^{-1}]\longrightarrow F_1^{(0)}\otimes F_2^{(0)}\otimes\mathbb{C}[T,T^{-1}]\longrightarrow F^{(0)}\longrightarrow\mathbb{C}[x]$ step by step. We will sketch the proof for a positive power of T, the proof in the other cases is similar.

$$\begin{aligned}
& T^kS_\mu(x^{(1)})S_\lambda(x^{(2)})\\
\longrightarrow\ & T^k\{\dots\psi_1(\lambda_{-p}-p)\psi_1^*(-p)\dots\psi_1(\lambda_0)\psi_1^*(0)\}\\
& \{\dots\psi_2(\mu_{-q}-q)\psi_2^*(-q)\dots\psi_2(\mu_0)\psi_2^*(0)\}\,|0\rangle\\
=\ & \{\dots\psi_1(\lambda_{-p}-p+k)\psi_1^*(-p+k)\dots\psi_1(\lambda_0+k)\psi_1^*(k)\}\\
& \{\dots\psi_2(\mu_{-q}-q-k)\psi_2^*(-q-k)\dots\psi_2(\mu_0-k)\psi_2^*(-k)\}\,T^k|0\rangle\\
=\ & \{\dots\psi_1(\lambda_{-p}-p+k)\psi_1^*(-p+k)\dots\psi_1(\lambda_0+k)\psi_1^*(k)\}\\
& \{\dots\psi_2(\mu_{-q}-q-k)\psi_2^*(-q-k)\dots\psi_2(\mu_0-k)\psi_2^*(-k)\}\\
& \{\psi_1(k)\psi_2^*(-k+1)\dots\psi_1(r)\psi_2^*(-r+1)\dots\psi_1(1)\psi_2^*(0)\}\,|0\rangle\\
=\ & \{\dots\psi_1(\lambda_{-p-k}-p)\psi_1^*(-p)\dots\psi_1(\lambda_{-k})\psi_1^*(0)\}\\
& \{\dots\psi_2(\mu_{-q}-q-k)\psi_2^*(-q-k)\dots\psi_2(\mu_0-k)\psi_2^*(-k)\}\\
& \{\psi_1(\lambda_0+k)\psi_2^*(-k+1)\dots\psi_1(\lambda_{-k+r}+r)\psi_2^*(-r+1)\dots\psi_1(\lambda_{-k+1}+1)\psi_2^*(0)\}\,|0\rangle\\
=\ & v_1(\lambda_{-k+1}+1)\wedge\dots\wedge v_1(\lambda_{-k+n_2}+n_2)\wedge v_1(\lambda_{-k})\wedge\dots\wedge v_1(\lambda_{-k-n_1+1}-n_1+1)\wedge\dots\\
& \wedge v_1(\lambda_{-l+1}+mn_2+1)\wedge\dots\wedge v_1(\lambda_0+k)\\
& \wedge v_2(\mu_0-k)\wedge\dots\wedge v_2(\mu_{-n_2+l+1}-(m+1)n_2+1)\\
& \wedge v_1(\lambda_{-k-mn_1}-mn_1)\wedge\dots\wedge v_1(\lambda_{-k-(m+1)n_1+1}-(m+1)n_1+1)\\
& \wedge v_2(\mu_{-n_2+l}-(m+1)n_2)\wedge\dots
\end{aligned}$$

Now putting the factors λ in decreasing order and using the boson-fermion correspondence σ we obtain formula (53). □

To illustrate the result one looks at a concrete example

Example 3.4. One considers here the simplest case $n=2$ and $n_1=n_2=1$. Now we get rid of the entier functions. $f_1(k)=2k-1$ and $f_2(k)=2k$, $k_1=k_2=1$. Then we get $(k>0)$:

$$\begin{aligned}
c_+ &= k^2-k\equiv 0 \bmod 2\\
\bar\gamma_k(r) &= 2\lambda_r+2k+r-1 && (-2k+2\le r\le 0)\\
\bar\gamma_k(2p+1) &= 2\lambda_{-k+p+1} && (p\le -k)\\
\bar\gamma_k(2(p+1)) &= 2\mu_{k+p+1} && (p<-k)\\
c_- &= k^2\equiv k \bmod 2\\
\bar\gamma_{-k}(r) &= 2\mu_r+2k+r && (-2k+1\le r\le 0)\\
\bar\gamma_{-k}(2p+1) &= 2\lambda_{k+p+1} && (p<-k)\\
\bar\gamma_{-k}(2(p+1)) &= 2\mu_{-k+p+1} && (p<-k)
\end{aligned}$$

We thus have

$\bar{\gamma}_k = (2\lambda_0 + 2k - 1, \ldots, 2\lambda_{-j} + 2k - j - 1, \ldots, 2\lambda_{-2k+1}, 2\mu_0, 2\lambda_{-2k}, \ldots, 2\mu_{-l}, 2\lambda_{-2k-l}, \ldots)$
$\bar{\gamma}_{-k} = (2\mu_0 + 2k, \ldots, 2\mu_{-j} + 2k - j, \ldots, 2\mu_{-2k+1} + 1, 2\mu_{-2k}, 2\lambda_0, \ldots, 2\mu_{-2k-l}, 2\lambda_{-l}, \ldots)$

In this case we can find a formula for the degree of this polynomial. Remember that on $\mathbb{C}[x]$ we have the principal degree deg. Then we easily derive that

(54) $$\deg(\gamma(T^k S_\mu(x^{(1)}) S_\lambda(x^{(2)}))) = 2|\mu| + 2|\lambda| + 2k^2 - k$$

for all $k \in \mathbb{Z}$. Here $|\mu| := \sum_{p\le 0} \mu_p$, the principal degree of S_μ. In the general case it is not so easy to find a nice expression for the degree of the image polynomial in terms of the principal degrees of the original Schur polynomials and the integer k. Therefore we first look at $\gamma(T^k)$. For $k > 0$ we have

(55) $$\bar{\gamma}_{k,0,0}(r) = f_1(k+r) - r = \left\lfloor \frac{k+r-1}{n_1} \right\rfloor n_2 + k$$

for $-k - mn_1 + 1 \le r \le 0$. For $k \le 0$ we get a similar formula. A straightforward calculation now gives:

(56) $$\deg(\gamma(T^k)) = \frac{nk(nk - n_1 n_2) + n_2^2 k_1 (n_1 - k_1) + n_1^2 k_2 (n_2 - k_2)}{2n_1 n_2}$$

This formula is valid for $k \in \mathbb{Z}$.

3.5. **The degree of a polynomial.** The principal degree of a polynomial in $\mathbb{C}[x]$ corresponds in $F^{(0)}$ with the eigenvalues of the operator H_0 where

(57) $$H_0 = \sum_{k\in\mathbb{Z}} k : \psi(k)\psi^*(k) :$$

This operator has nonnegative integer eigenvalues. It is an element of $\pi(\overline{gl_\infty})$ namely $H_0 = \pi(\sum_{k\in\mathbb{Z}} k\mathcal{E}_{kk})$. We have $\alpha(k) = \sum_{l\in\mathbb{Z}} : \psi(l)\psi^*(l+k) :$. These $\alpha(k)$ are bosonic oscillators. The commutator of $\alpha(k)$ with H_0 gives: $[H_0, \alpha(k)] = -k\alpha(k)$. This means that we can assign a degree $-k$ to $\alpha(k)$, because $\alpha(k)$ changes the eigenvalue of H_0 with $-k$. The principal degree of x_k can then be computed using the isomorphism between $\mathbb{C}[x]$ and $F^{(0)}$. This isomorphism gives $\deg(x_k) = k$. We can also look at the oscillators $\alpha_i(k)$ which are defined by

(58) $$\alpha_i(k) := \sum_{l\in\mathbb{Z}} : \psi_i(l)\psi_i^*(l+k) :$$

Now we can try to compute the commutator of H_0 with $\alpha_i(k)$. Using

$$H_0 = \sum_{j=1}^{2} \sum_{p\in\mathbb{Z}} f_j(p) : \psi_j(p)\psi_j^*(p) :$$

we get the following formula

(59) $$[H_0, \alpha_i(k)] = \sum_{p\in\mathbb{Z}} (f_i(p) - f_i(p+k))\psi_i(p)\psi_i^*(p+k)$$

The right hand side is in general not proportional to $\alpha_i(k)$, so we cannot assign a principal degree to $\alpha_i(k)$. If $k = n_i l$ then we can assign a degree to it because then $[H_0, \alpha_i(n_i l)] = -nl\alpha_i(n_i l)$. So $\deg(\alpha_i(n_i l)) = -nl$ and $\deg(x^{(i)}_{n_i l}) = nl$. In the example we have $n_1 = n_2 = 1$, so every k is of the form $k = n_i l$. So in this case we have $\deg(\alpha_i(k)) = -2k$ and $\deg(x^{(i)}_k) = 2k$.

In [4] another degree has been used. That degree corresponds with the eigenvalues of the operator D_0 defined by

$$D_0 := \sum_{p>0}\sum_{i=1}^{2}\frac{1}{n_i}\alpha_i(-p)\alpha_i(p)+\frac{1}{2}\sum_{i=1}^{2}\frac{1}{n_i}\alpha_i(0)^2+\frac{1}{2}|H_n|^2 I \tag{60}$$

$$= \sum_{i=1}^{2}\sum_{p\in\mathbb{Z}}\frac{1}{n_i}p:\psi_i(p)\psi_i^*(p):-\frac{1}{2}\sum_{i=1}^{2}\frac{1}{n_i}\alpha_i(0)+\frac{1}{2}|H_n|^2 I \tag{61}$$

Not all eigenvalues of D_0 are integers. Therefore we multiply the eigenvalues with a constant N. Call the corresponding degree $\deg_0$. Then you can assign a degree $\deg_0$ to $\alpha_i(k)$: $\deg_0(\alpha_i(k)) = -N\frac{k}{n_i}$ and $\deg_0(x_k^{(i)}) = N\frac{k}{n_i}$. It is not a surprise that in general one cannot assign a degree $\deg_0$ to $\alpha(k)$. One only has $\deg_0(\alpha(nl)) = -Nl$ and $\deg_0(x_{nl}) = Nl$. In the next subsections we consider the homogeneous realization.

3.6. **The case 3=1+1+1.** In this subsection we present some formulas for the isomorphism γ in the case $3 = 1+1+1$. In this case we have the two operators T_1 and T_2 which anticommute.

Theorem 3.7. *For $k_2 \geq 2k_1 \geq 0$ we have*

$$\begin{aligned}\gamma(T_1^{k_1}T_2^{k_2}) &= (-1)^{\frac{1}{2}k_1(k_1-1)+k_1(k_2-k_1)}S_{\tilde{\gamma}}(x)\\ &= (-1)^{\frac{1}{2}k_1(k_1-1)+\frac{1}{2}(k_2-k_1)(3k_2+k_1-3)}S_{\bar{\gamma}}(x)\end{aligned}$$

where

$$\begin{array}{rcll}\tilde{\gamma}(3p) &=& -6p+1 & 0\geq p>-k_1\\ \tilde{\gamma}(3p) &=& -6p-3k_1+2 & -k_1\geq p>-k_2\\ \tilde{\gamma}(q) &=& 0 & \textit{otherwise}\\ \bar{\gamma}(p) &=& 3(k_2-k_1)-1+2p & 0\geq p>-k_2+2k_1\\ \bar{\gamma}(-k_2+2k_1+2p) &=& k_2+k_1-1+p & 0\geq p>-k_2-k_1\\ \bar{\gamma}(-k_2+2k_1+2p-1) &=& k_2+k_1-1+p & 0\geq p>-k_2-k_1\\ \bar{\gamma}(q) &=& 0 & \textit{otherwise}\end{array}$$

$$\deg(\gamma(T_1^{k_1}T_2^{k_2})) = 3k_1^2+3k_2^2-3k_1k_2-k_1-k_2$$

Proof. $T_1^{k_1}T_2^{k_2}|0\rangle = (-1)^{\frac{1}{2}k_1(k_1-1)}(T_1T_2)^{k_1}T^{k_2-k_1}|0\rangle$.

$$\begin{aligned}&(T_1T_2)^{k_1}T^{k_2-k_1}|0\rangle\\ =\ &(T_1T_2)^{k_1}\left(\psi_2(k_2-k_1)\psi_3^*(-k_2+k_1+1)\dots\psi_2(1)\psi_3^*(0)\right)|0\rangle\\ =\ &(-1)^{k_1(k_2-k_1)}\left(\psi_2(k_2-k_1)\psi_3^*(-k_2+1)\dots\psi_2(1)\psi_3^*(-k_1)\right)\\ &\left(\psi_1(k_1)\psi_3^*(-k_1+1)\dots\psi_1(1)\psi_3^*(0)\right)|0\rangle\\ =\ &(-1)^{k_1(k_2-k_1)}v_1(1)\wedge v_2(0)\wedge v_1(0)\wedge\dots\wedge v_1(k_1)\wedge v_2(-k_1+1)\wedge v_1(-k_1+1)\wedge\\ &v_2(1)\wedge v_2(-k_1)\wedge v_1(-k_1)\wedge\dots\wedge v_2(k_2-k_1)\wedge v_2(-k_2+1)\wedge v_1(-k_2+1)\wedge\\ &v_3(-k_2)\wedge v_2(-k_2)\wedge v_1(-k_2)\wedge\dots\\ \longrightarrow\ &(-1)^{k_1(k_2-k_1)}S_{\tilde{\gamma}}(x)\end{aligned}$$

Putting the $v_i(k)$ into decreasing order we find the second formula. The degree is found summing the indices of the Schur polynomial $S_{\bar{\gamma}}$. □

These formulas are only valid for $k_2 \geq 2k_1 \geq 0$. It is easy to calculate the formulas in all other cases, but that is left to the reader. The formulas for the degree of $T_1^{k_1}$ and $T_2^{k_2}$ can easily be calculated using the isomorphism γ. We then obtain

$$\deg(\gamma(T_1^{k_1})) = 3k_1^2 - k_1 \text{ and } \deg(\gamma(T_2^{k_2})) = 3k_2^2 - k_2.$$

These formulas are valid for $k_1, k_2 \in \mathbb{Z}$.

3.8. **Principal degree in homogeneous realization.** In previous sections we generalized the case $1+1=2$ to $n_1+n_2=n$. In this section we generalize it to $1+1+\ldots+1=n$, the homogeneous realization. Here we have the isomorphism

$$\mathbb{C}[x^{(1)}, \ldots, x^{(n)}] \otimes \mathbb{C}[\hat{T}_n] \simeq (\bigotimes_{i=1}^{n} F_i^{(0)}) \otimes \mathbb{C}[\hat{T}_n] \simeq F^{(0)} \simeq \mathbb{C}[x] \tag{62}$$

Here $\hat{T}_n$ is a group of operators T_i, $1 \leq i \leq n-1$. T_i replaces a fermion of type $i+1$ by a fermion of type i. They satisfy the relations $T_iT_j = -T_jT_i$ when $|i-j| = 1$, and $T_iT_j = T_jT_i$ otherwise, cf. [4].

It is easy to find an expression for the image of a product of Schur-functions:

$$\gamma\left(\prod_{i=1}^{n} S_{\lambda^{(i)}}(x^{(i)})\right) = S_{\bar{\gamma}}(x) \tag{63}$$

$$\bar{\gamma}(mn+i) = n\lambda_{m+1}^{(i)} \tag{64}$$

so

$$\bar{\gamma} = (n\lambda_0^{(n)}, \ldots, n\lambda_0^{(1)}, n\lambda_{-1}^{(n)}, \ldots, n\lambda_{-1}^{(1)}, \ldots)$$

We thus have

$$\deg\left(\gamma(\prod_{i=1}^{n} S_{\lambda^{(i)}}(x^{(i)}))\right) = n\sum_{i=1}^{n} |\lambda^{(i)}| \tag{65}$$

On $F^{(0)}$ $\alpha(0)$ acts as 0. Because $\alpha(0) = \sum \alpha_i(0)$ we can eliminate $\alpha_n(0)$. Introduce now

$$\beta_i := \sum_{j=1}^{i} \alpha_j(0) \quad 1 \leq i \leq n-1 \tag{66}$$

We then have $\alpha_1(0) = \beta_1$, $\alpha_i(0) = \beta_i - \beta_{i-1}$ $(1 < i < n)$ and $\alpha_n(0) = -\beta_{n-1}$. The eigenvalue of $\alpha_i(0)$ is the charge of the fermions of type i. The eigenvalue of β_i corresponds with the power of T_i. The next proposition and corollary show that in the homogeneous case we can express the principal degree of the image polynomial in terms of the principal degrees of the original Schur polynomials and the powers of the operators T_i.

Proposition 3.9. *On $F^{(0)}$ we have the following equalities:*

$$H_0 = n\sum_{i=1}^{n}\sum_{k>0} \alpha_i(-k)\alpha_i(k) + n\sum_{i=1}^{n-1} \alpha_i(0)^2 + n\sum_{i,j=1,i<j}^{n-1} \alpha_i(0)\alpha_j(0) - \sum_{i=1}^{n-1}(n-i)\alpha_i(0) \tag{67}$$

$$= n\sum_{i=1}^{n}\sum_{k>0} \alpha_i(-k)\alpha_i(k) + n\sum_{i=1}^{n-1} \beta_i^2 - n\sum_{i=2}^{n-1} \beta_i\beta_{i-1} - \sum_{i=1}^{n-1} \beta_i \tag{68}$$

Proof.

$$\begin{aligned}
H_0 &= \sum_{k\in\mathbb{Z}} k : \psi(k)\psi^*(k) : \\
&= \sum_{i=1}^{n}\sum_{k\in\mathbb{Z}} (n(k-1)+i) : \psi_i(k)\psi_i^*(k) : \\
&= n\sum_{i=1}^{n} H_0^{(i)} + \sum_{i=1}^{n}(i-n)\alpha_i(0)
\end{aligned}$$

Now we use $H_0^{(i)} = \sum_{k>0}\alpha_i(-k)\alpha_i(k) + \frac{1}{2}\alpha_i(0)^2 + \frac{1}{2}\alpha_i(0)$. Then we obtain

$$\begin{aligned}
H_0 &= n\sum_{i=1}^{n}\sum_{k>0}\alpha_i(-k)\alpha_i(k) + \frac{1}{2}n\sum_{i=1}^{n}(\alpha_i(0)^2+\alpha_i(0)) + \sum_{i=1}^{n}(i-n)\alpha_i(0) \\
&= n\sum_{i=1}^{n}\sum_{k>0}\alpha_i(-k)\alpha_i(k) + \frac{1}{2}n\sum_{i=1}^{n-1}\alpha_i(0)^2 + \frac{1}{2}n\sum_{i,j=1}^{n-1}\alpha_i(0)\alpha_j(0) - \sum_{i=1}^{n-1}(n-i)\alpha_i(0) \\
&= n\sum_{i=1}^{n}\sum_{k>0}\alpha_i(-k)\alpha_i(k) + n\sum_{i=1}^{n-1}\alpha_i(0)^2 + n\sum_{i,j=1,i<j}^{n-1}\alpha_i(0)\alpha_j(0) - \sum_{i=1}^{n-1}(n-i)\alpha_i(0) \\
&= n\sum_{i=1}^{n}\sum_{k>0}\alpha_i(-k)\alpha_i(k) + \frac{1}{2}n(\beta_1^2 + \sum_{i=2}^{n-1}(\beta_i-\beta_{i-1})^2 + \beta_{n-1}^2) - \sum_{i=1}^{n-1}\beta_i \\
&= n\sum_{i=1}^{n}\sum_{k>0}\alpha_i(-k)\alpha_i(k) + n\sum_{i=1}^{n-1}\beta_i^2 - n\sum_{i=2}^{n-1}\beta_i\beta_{i-1} - \sum_{i=1}^{n-1}\beta_i
\end{aligned}$$

□

Corollary 3.10.

$$\deg\left(\gamma\left(T_1^{k_1}\dots T_{n-1}^{k_{n-1}}\prod_{i=1}^{n}S_{\lambda^{(i)}}(x^{(i)})\right)\right) = n\sum_{i=1}^{n}|\lambda^{(i)}| + n\sum_{i=1}^{n-1}k_i^2 - n\sum_{i=2}^{n-1}k_ik_{i-1} - \sum_{i=1}^{n-1}k_i \tag{69}$$

In corollary 3.10 we see that deg mod$n$ only is determined by the k_i and not by the partitions $\lambda^{(i)}$. We have already seen the formula for $n=2$ in equation (54).

Example 3.11. For $n=3$ corollary 3.10 gives:

$$\deg\left(\gamma\left(T_1^{k_1}T_2^{k_2}S_{\lambda^{(1)}}(x^{(1)})S_{\lambda^{(2)}}(x^{(2)})\right)\right) = 3k_1^2 - k_1 + 3k_2^2 - k_2 - 3k_1k_2 \tag{70}$$

This formula agrees with the formulas (3.6) and proposition 3.7.

According to corollary 3.10 we have $\deg(x_k^{(i)}) = nk$. In this homogeneous case the degree $\deg_0$ is almost equal to the principal degree deg. It is easy to show that $\deg_0(x_k^{(i)}) = k$.

3.12. **q-dimensions.** We have the isomorphism

$$F^{(0)} \simeq \left(\bigotimes_{i=1}^{n} F_i^{(0)}\right)\otimes\mathbb{C}\hat{T} \tag{71}$$

We can now take q-dimensions of equation (71) with respect to the principal degree. Using $\varphi(q) = \prod_{i=1}^{\infty}(1-q^i)$ we obtain the following formula

$$\text{trace}_{F(0)} q^{H_0} = \frac{1}{\varphi(q)} = \left(\frac{1}{\varphi(q^n)}\right)^n \sum_{k_1,\dots,k_{n-1}\in\mathbb{Z}} q^{n\sum k_i^2 - n\sum k_i k_{i-1} - \sum k_i} \tag{72}$$

We can rewrite the second part of the equation. We then get

$$\sum_{k_1,\dots,k_{n-1}\in\mathbb{Z}} q^{n\sum k_i^2 - n\sum k_i k_{i-1} - \sum k_i} = \frac{\varphi(q^n)^n}{\varphi(q)} \tag{73}$$

Example 3.13. For $n = 2$ equation (73) becomes

$$\sum_{k\in\mathbb{Z}} q^{2k^2-k} = \frac{\varphi(q^2)^2}{\varphi(q)} = \frac{\prod_{j=1}^{\infty}(1-q^{2j})}{\prod_{k=1}^{\infty}(1-q^{2k-1})} \tag{74}$$

The left hand side is equal to $\sum_{k\geq 0} q^{\frac{1}{2}k(k+1)}$, so we get

$$\sum_{k\geq 0} q^{\frac{1}{2}k(k+1)} = \frac{\varphi(q^2)^2}{\varphi(q)} \tag{75}$$

The formulas (74) and (75) can also be derived from the classical Jacobi triple product identity (see [3]):

$$\prod_{k\geq 1}(1-u^{k-1}v^k)(1-u^k v^{k-1})(1-u^k v^k) = \sum_{j\in\mathbb{Z}}(-1)^j u^{\frac{1}{2}j(j+1)} v^{\frac{1}{2}j(j-1)} \tag{76}$$

Take $u = -q^3$ and $v = -q$. Then we get

$$\begin{aligned}
\sum_{j\in\mathbb{Z}} q^{2j^2+j} &= \prod_{k\geq 1}(1+q^{4k-3})(1+q^{4k-1})(1-q^{4k}) \\
&= \prod_{k\geq 1}(1+q^{4k-3})(1+q^{4k-1})(1+q^{2k})(1-q^{2k}) \\
&= \prod_{k\geq 1}(1+q^k)(1-q^{2k}) \\
&= \prod_{k\geq 1}\frac{(1-q^{2k})^2}{1-q^k} = \frac{\varphi(q^2)^2}{\varphi(q)}
\end{aligned}$$

so equation (74) is verified. For equation (75) we can take $u = -q$ and $v = -1$ in the Jacobi identity (76).

3.14. **Some formulas on $F^{(k)}$.** Because of the isomorphism $F^{(k)} \simeq F^{(0)}$ we can also look at the principal degree on $F^{(k)}$. We can write this in terms of the m_i, the eigenvalues of the $\alpha_i(0)$, or in terms of the k_i, the eigenvalues of the β_i. We have $k_i = \sum_{j=1}^{i} m_j$. Note that we know have $\sum \alpha_i = k$. We then obtain for the principal degree on $F^{(k)}$ the following formulas:

$$\deg\left(\gamma\left(T_1^{k_1}\dots T_{n-1}^{k_{n-1}}\prod_{i=1}^{n} S_{\lambda^{(i)}}(x^{(i)})\right)\right) =$$

(77) $$= n\sum_{i=1}^{n}|\lambda^{(i)}| + \frac{1}{2}n\sum_{i=1}^{n} m_i^2 + \sum_{i=1}^{n} im_i - \frac{1}{2}nk$$

(78) $$= n\sum_{i=1}^{n}|\lambda^{(i)}| + n\sum_{i=1}^{n-1} k_i^2 - n\sum_{i=2}^{n-1} k_ik_{i-1} - \sum_{i=1}^{n-1} k_i - nkk_{n-1} + \frac{1}{2}nk(k+1)$$

This leads to the following q-dimension formulas:

(79) $$\mathrm{trace}_{F^{(k)}} q^{H_0} = \frac{q^{\frac{1}{2}k(k+1)}}{\varphi(q)}$$

(80) $$= \frac{q^{-\frac{1}{2}nk}}{\varphi(q^n)^n} \sum_{m_1,\dots,m_n\in\mathbb{Z},\sum m_i=k} q^{\frac{1}{2}n\sum m_i^2+\sum im_i}$$

(81) $$= \frac{q^{\frac{1}{2}nk(k+1)}}{\varphi(q^n)^n} \sum_{k_1,\dots,k_{n-1}\in\mathbb{Z}} q^{n\sum k_i^2 - n\sum k_ik_{i-1} - nkk_{n-1} - \sum k_i}$$

We can rewrite this formulas to

(82) $$\frac{\varphi(q^n)^n}{\varphi(q)} = \sum_{m_i\in\mathbb{Z},\sum m_i=k} q^{\frac{1}{2}n\sum(m_i-\frac{k}{n})^2+\sum i(m_i-\frac{k}{n})}$$

(83) $$= \sum_{m_i\in-\frac{k}{n}+\mathbb{Z},\sum m_i=0} q^{\frac{1}{2}n\sum m_i^2+\sum im_i}$$

and

(84) $$\frac{\varphi(q^n)^n}{\varphi(q)} = \sum_{k_i\in\mathbb{Z}} q^{n\sum(k_i-i\frac{k}{n})^2 - n\sum(k_i-i\frac{k}{n})(k_{i-1}-(i-1)\frac{k}{n}) - \sum(k_i-i\frac{k}{n})}$$

(85) $$= \sum_{k_i\in-i\frac{k}{n}+\mathbb{Z}\ (1\le i<n)} q^{n\sum k_i^2 - n\sum k_ik_{i-1} - \sum k_i}$$

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